

A Model of Quark and Lepton Mixing and Mass Hierarchy

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Abstract

It is shown that an idea proposed in 1996 that relates in a qualitatively correct way the inter-family mass hierarchies of the up quarks, down quarks, charged leptons, and neutrinos, can be combined with a predictive scheme recently proposed for relating quark mixing and neutrino mixing. In the resulting model, the entire flavor structure of the quarks and leptons is expressible in terms of two “master matrices”: a diagonal matrix that gives the inter-family mass ratios, and an off-diagonal matrix that controls all flavor mixing.

1 Introduction

The flavor problem has two aspects: explaining the pattern of quark and lepton mixing angles and explaining the pattern of inter-family mass hierarchies. In this paper we show that an idea proposed in 1996 [1] for explaining the mass hierarchies can be successfully combined with an idea proposed in 2012 [2] for explaining the mixing. We shall refer to these as the “BB idea” and the “BC idea” respectively. The two ideas are actually complementary, and by combining them a model emerges that is simpler and more explanatory than either by itself. We shall first briefly review the two ideas and then show how they can be combined.

The BB idea was based on the observation that the inter-family mass hierarchy of the up quarks (u, c, t) is stronger than those of the down quarks (d, s, b) and charged leptons (e, μ, τ), which in turn are stronger than that of the neutrinos. Ref. [1] pointed out that the strengths of these hierarchies correlate, in an $SU(5)$ framework, with the number of fermion 10-plets that appear in the corresponding Yukawa terms. Up quark masses come from $(\mathbf{10} \mathbf{10})\mathbf{5}_H$ terms, which have two factors of fermion 10-plets. Down quark and charged lepton masses come from $(\mathbf{10} \mathbf{\bar{5}})\mathbf{\bar{5}}_H$ terms, which have only one such factor. And the neutrino masses come from effective dimension-5 $(\mathbf{\bar{5}} \mathbf{\bar{5}})\mathbf{5}_H\mathbf{5}_H$ terms, which contain no such factors.

The BB idea was that every fermion 10-plet in a Yukawa term is accompanied by a factor in the mass matrix of a hierarchical, diagonal matrix H , which one can write as $H = \text{diag}(\alpha, \beta, 1)h$, where $\alpha \ll \beta \ll 1$. This can happen as the result of the mixing of the 10-plets in the usual three chiral families, which we denote by $\mathbf{10}_i^0 + \mathbf{\bar{5}}_i^0$, with “extra” vectorlike 10-plets, which we denote by $\mathbf{10}'_i + \mathbf{\bar{10}}'_i$

($i = 1, 2, 3$). Let the “underlying” Yukawa terms that give electroweak-breaking quark and lepton masses be of the form

$$(\mathbf{10}_i^0 Y_{ij}^u \mathbf{10}_j^0) \mathbf{5}_H + (\mathbf{10}_i^0 Y_{ij} \bar{\mathbf{5}}_j^0) \bar{\mathbf{5}}_H + (\mathbf{10}_i^0 y_{ij} \bar{\mathbf{5}}_j^0) \overline{\mathbf{45}}_H + (\bar{\mathbf{5}}_i^0 Y_{ij}^\nu \bar{\mathbf{5}}_j^0) \mathbf{5}_H \mathbf{5}_H / M_R. \quad (1)$$

The role of the term with the $\mathbf{45}_H$ of Higgs fields is to give different contributions to the mass matrices of the down quarks and charged leptons [3] and thus avoid the “bad” predictions of minimal $SU(5)$ that $m_e = m_d$, and $m_\mu = m_s$ at the GUT scale. Suppose that $\mathbf{10}_i^0$ and $\mathbf{10}_i'$ mix in a family-diagonal way to produce a light linear combination $\mathbf{10}_i$ that contains Standard Model fermions and an orthogonal linear combination $\mathbf{10}_i^h$ that is superheavy. Then one can write

$$\mathbf{10}_i^0 = \cos \theta_i \mathbf{10}_i + \sin \theta_i \mathbf{10}_i^h. \quad (2)$$

Substituting this into Eq. (1), one obtains for the effective Yukawa terms of the Standard Model fermions

$$(\mathbf{10}_i \cos \theta_i Y_{ij}^u \cos \theta_j \mathbf{10}_j) \mathbf{5}_H + (\mathbf{10}_i \cos \theta_i Y_{ij} \bar{\mathbf{5}}_j^0) \bar{\mathbf{5}}_H + (\mathbf{10}_i \cos \theta_i y_{ij} \bar{\mathbf{5}}_j^0) \overline{\mathbf{45}}_H + (\bar{\mathbf{5}}_i^0 Y_{ij}^\nu \bar{\mathbf{5}}_j^0) \mathbf{5}_H \mathbf{5}_H / M_R. \quad (3)$$

Therefore, the effective quark and lepton mass terms of the Standard Model quarks and leptons can be written

$$\begin{aligned} M_u &= H m_u H, \\ M_d &= H m_d, \\ M_\ell &= m_\ell H, \\ M_\nu &= m_\nu, \end{aligned} \quad (4)$$

where

$$H = \begin{pmatrix} \cos \theta_1 & 0 & 0 \\ 0 & \cos \theta_2 & 0 \\ 0 & 0 & \cos \theta_3 \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} h, \quad (5)$$

and where $(m_u)_{ij} = Y_{ij}^u v_5$, $(m_d)_{ij} = Y_{ij} v_{\bar{\mathbf{5}}} + y_{ij} v_{\overline{\mathbf{45}}}$, $(m_\ell)_{ij} = Y_{ij} v_{\bar{\mathbf{5}}} - 3y_{ij} v_{\overline{\mathbf{45}}}$, and $(m_\nu)_{ij} = Y_{ij}^\nu (v_5^2 / M_R)$. These four “underlying” mass matrices m_u, m_d, m_ℓ , and m_ν are not assumed to have any special form, and therefore for each of them one expects all the elements to be roughly of the same order. From Eqs. (4) and (5) one has

$$\begin{aligned} M_u &\sim \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha \\ \alpha\beta & \beta^2 & \beta \\ \alpha & \beta & 1 \end{pmatrix} \mu_u, & M_\nu &\sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \mu_\nu \\ M_d &\sim \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ 1 & 1 & 1 \end{pmatrix} \mu_d, & M_\ell &\sim \begin{pmatrix} \alpha & \beta & 1 \\ \alpha & \beta & 1 \\ \alpha & \beta & 1 \end{pmatrix} \mu_\ell, \end{aligned} \quad (6)$$

where “ \sim ” means that the various elements are of the given order of magnitude. This obviously gives

$$\begin{aligned} m_u : m_c : m_t &\sim \alpha^2 : \beta^2 : 1 \\ m_d : m_s : m_b &\sim \alpha : \beta : 1 \\ m_e : m_\mu : m_\tau &\sim \alpha : \beta : 1 \\ m_{\nu_1} : m_{\nu_2} : m_{\nu_3} &\sim 1 : 1 : 1. \end{aligned} \quad (7)$$

This reproduces well, in a qualitative way, the strengths of the inter-family mass hierarchies of the different types of fermions. Also from inspection of Eq. (6) it is apparent that

$$U_{MNS} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad V_{CKM} \sim \begin{pmatrix} 1 & \alpha/\beta & \alpha \\ \alpha/\beta & 1 & \beta \\ \alpha & \beta & 1 \end{pmatrix}. \quad (8)$$

This gives $O(1)$ MNS mixing angles and small CKM mixing angles, with $|V_{ub}| \sim |V_{us}V_{cb}|$, which also is qualitatively correct. On the other hand, since there are no constraints on the forms of the four underlying 3×3 mass matrices m_u, m_d, m_ℓ , and m_ν , the BB idea in this form has many free parameters and can only make qualitative post-dictions rather than precise quantitative predictions.

We turn now to a review of the BC idea [2]. The BC idea is that all inter-family mixing among the Standard Model fermions arises from a single source. This source is the mixing between the $\bar{\mathbf{5}}$ multiplets in the three chiral families and those in extra $\bar{\mathbf{5}} + \mathbf{5}$ vectorlike pairs. Because mixing comes from the $\bar{\mathbf{5}}$ multiplets of $SU(5)$, there is large mixing only for left-handed leptons and right-handed quarks, thus also explaining why the MNS mixing is large and the CKM mixing is small. (This is the basic idea of so-called “lopsided models” [1, 4].)

The specific assumption in the BC model is that if there existed *only* the three chiral families, then the quark and lepton mass matrices would be diagonal due to abelian family symmetries. But the extra $\bar{\mathbf{5}} + \mathbf{5}$ vectorlike multiplets, which are assumed *not* to transform under the family symmetries, are able to mix with the chiral families due to spontaneous breaking of those symmetries. This induces mixing among the families of Standard Model quarks and leptons.

In the BC model, the fermion content consists of $(\mathbf{10}_i^0 + \bar{\mathbf{5}}_i^0) + (\bar{\mathbf{5}}'_A + \mathbf{5}'_A)$, where $i = 1, 2, 3$ and $A = 1, 2, \dots, N$. There are abelian symmetries $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$, such that under $Z_2^{(j)}$ the fields $\mathbf{10}_i$ and $\bar{\mathbf{5}}_i$ are odd if $i = j$ but even if $i \neq j$. As a consequence, the “underlying” Yukawa terms of the three chiral families have the family-diagonal form

$$(\mathbf{10}_i^0 Y_i^u \mathbf{10}_i^0) \mathbf{5}_H + (\mathbf{10}_i^0 Y_i \bar{\mathbf{5}}_i^0) \bar{\mathbf{5}}_H + (\mathbf{10}_i^0 y_i \bar{\mathbf{5}}_i^0) \bar{\mathbf{4}}\mathbf{5}_H + (\bar{\mathbf{5}}_i^0 Y_i^\nu \bar{\mathbf{5}}_i^0) \mathbf{5}_H \mathbf{5}_H / M_R. \quad (9)$$

The abelian family symmetries are broken spontaneously by the VEVs of Standard-Model-singlet scalars $\mathbf{1}_{Hi}$, $i = 1, 2, 3$. (Like the fermions, $\mathbf{1}_{Hi}$ is odd under $Z_2^{(j)}$ if $i = j$ and even otherwise.) This allows the chiral fermions and the extra vectorlike fermions to mix through the following Yukawa terms: $y_{iA}(\bar{\mathbf{5}}_i^0 \mathbf{5}'_A)(\mathbf{1}_{Hi}) + y'_{AB}(\bar{\mathbf{5}}'_A \mathbf{5}'_B)(\mathbf{1}_H)$. Because of these terms, the $\bar{\mathbf{5}}_i^0$ and $\bar{\mathbf{5}}'_A$ mix to produce three linear combinations, $\bar{\mathbf{5}}_i$, which contain Standard Model quarks and leptons, and N linear combinations $\bar{\mathbf{5}}_A^h$ that are orthogonal to them and superheavy. One can therefore write

$$\bar{\mathbf{5}}_i^0 = A_{ij} \bar{\mathbf{5}}_j + B_{iA} \bar{\mathbf{5}}_A^h, \quad (10)$$

where A and B are non-diagonal matrices satisfying $AA^\dagger + BB^\dagger = I$. Substituting Eq. (10) into Eq. (9), one finds that the effective mass terms for the Standard Model quarks and leptons can be written

$$\begin{aligned} M_u &= m_u, \\ M_d &= m_d A, \\ M_\ell &= A^T m_\ell, \\ M_\nu &= A^T m_\nu A, \end{aligned} \tag{11}$$

where $(m_u)_{ij} = \delta_{ij} Y_i^u v_5$, $(m_d)_{ij} = \delta_{ij} (Y_i v_5 + y_i v_{45})$, $(m_\ell)_{ij} = \delta_{ij} (Y_i v_5 - 3y_i v_{45})$, and $(m_\nu)_{ij} = \delta_{ij} Y_i^\nu (v_5^2/M_R)$. The non-zero elements of these diagonal matrices are free parameters of the model. To fit the observed quark and lepton masses they must be hierarchical, but the BC model does not attempt to explain these hierarchies or relate them to each other.

As shown in [2], the matrix A can be brought by changes of basis to the form

$$A = \mathcal{D} A_\Delta \mathcal{U}, \tag{12}$$

where \mathcal{U} is a unitary matrix, \mathcal{D} is the diagonal matrix $\text{diag}(\delta, \epsilon, 1)d$, and A_Δ is a triangular matrix of the form

$$A_\Delta = \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

and a , b , and c are real numbers. The matrix \mathcal{U} can be eliminated by redefining the fields of the right-handed quarks and left-handed leptons. These redefinitions do not affect the CKM or MNS mixing matrices. (They do not affect the CKM matrix, as only a redefinition of the *right-handed* quarks is involved. They do not affect the MNS mixing matrix, since the same redefinition is done on the neutrinos and charged leptons.) The diagonal matrix \mathcal{D} can be absorbed into the diagonal matrices: $\overline{m}_u \equiv m_u$, $\overline{m}_d \equiv m_d \mathcal{D}$, $\overline{m}_\ell \equiv m_\ell \mathcal{D}$, and $\overline{m}_\nu = m_\nu \mathcal{D}^2$. After these redefinitions one has

$$\begin{aligned} M_u &= \overline{m}_u, \\ M_d &= \overline{m}_d A_\Delta, \\ M_\ell &= A_\Delta^T \overline{m}_\ell, \\ M_\nu &= A_\Delta^T \overline{m}_\nu A_\Delta. \end{aligned} \tag{14}$$

The elements of the diagonal matrix \overline{m}_u are obviously just the masses of the up quarks, $\overline{m}_u = \text{diag}(m_u, m_c, m_t)$. One can also easily show that to a very good approximation the elements of the diagonal matrices \overline{m}_d and \overline{m}_ℓ are the masses of down quarks and charged leptons: $\overline{m}_d \cong \text{diag}(m_d, m_s, m_b)$, and $\overline{m}_\ell \cong \text{diag}(m_e, m_\mu, m_\tau)$. That means that in the basis in which the up quark mass matrix is diagonal the mass matrix of the down quarks is given by

$$M_d = \begin{pmatrix} m_d & m_d b & m_d c e^{i\theta} \\ 0 & m_s & m_s a \\ 0 & 0 & m_b \end{pmatrix}. \tag{15}$$

From this one can read off directly that $|V_{us}| \cong \frac{m_d}{m_s} b$, $|V_{cb}| \cong \frac{m_s}{m_b} a$, and $V_{ub} \cong \frac{m_d}{m_b} c e^{i\theta}$. Therefore, the triangular matrix A_Δ can be written

$$A_\Delta = \begin{pmatrix} 1 & \left| \frac{m_s}{m_d} V_{us} \right| & \left| \frac{m_b}{m_d} V_{ub} \right| e^{i\delta_{KM}} \\ 0 & 1 & \left| \frac{m_b}{m_s} V_{cb} \right| \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

The mass matrix $M_\ell = A_\Delta^T \overline{m}_\ell$ of the charged leptons, given in Eq. (14), is obviously not diagonal. However, it is easily seen that it is diagonalized by rotations of the *right-handed* leptons by angles that are of order m_e/m_μ , m_μ/m_τ , and m_e/m_τ , while the required rotations of the *left-handed* leptons are only of order the squares of these ratios, and thus negligible. As far as the left-handed charged leptons are concerned, therefore, we are effectively in the mass basis. Combining Eq. (14) and (16), one has the following expression for the neutrino mass matrix

$$M_\nu = \mu_\nu \begin{pmatrix} 1 & 0 & 0 \\ \left| \frac{m_s}{m_d} V_{us} \right| & 1 & 0 \\ \left| \frac{m_b}{m_d} V_{ub} \right| e^{i\delta_{KM}} & \left| \frac{m_b}{m_s} V_{cb} \right| & 1 \end{pmatrix} \begin{pmatrix} qe^{i\theta_q} & 0 & 0 \\ 0 & pe^{i\theta_p} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \left| \frac{m_s}{m_d} V_{us} \right| & \left| \frac{m_b}{m_d} V_{ub} \right| e^{i\delta_{KM}} \\ 0 & 1 & \left| \frac{m_b}{m_s} V_{cb} \right| \\ 0 & 0 & 1 \end{pmatrix}. \quad (17)$$

The five free parameters p , θ_p , q , θ_q , and μ_ν determine all nine neutrino properties: the three neutrino masses, three MNS mixing angles, the Dirac CP phase of the leptons, and the two Majorana CP phases. So there are four predictions. These are discussed in detail in [2]. What was found there was that a good fit is obtained to the three neutrino mixing angles and to the neutrino mass splittings, and that the Dirac CP phase of the leptons is predicted to be roughly 210 degrees. The values of the parameters that gave the best fits were $p = 0.1525$, $q = 0.01405$, $\theta_p = -2.73$ radians, and $\theta_q = -0.352$ radians. The values of p and q are important for our later discussions.

2 Combining the two ideas

In the BB idea, all the inter-family mass hierarchies come from the single matrix H , while in the BC idea all the inter-family mixing comes from the single matrix A . The question naturally arises whether these two ideas can be combined in such a way that the whole flavor structure can be accounted for with *only* the matrices A and H , thereby producing a more predictive and explanatory model. The answer is yes, as we shall now show by describing a specific model that does this.

The fermion content of the model consists of the following $SU(5)$ multiplets:

$$(\mathbf{10}_i^0 + \overline{\mathbf{5}}_i^0)_{i=1,2,3} + (\mathbf{10}'_A + \overline{\mathbf{10}}'_A)_{A=1,2,3} + (\overline{\mathbf{5}}'_m + \mathbf{5}'_m)_{m=1,2,\dots,N}. \quad (18)$$

Yukawa terms involving only $\mathbf{10}_i^0$ and $\overline{\mathbf{5}}_i^0$ will give rise to “underlying” mass matrices that get multiplied by factors of the matrices H and A . In order for H and A to account for *all* the flavor structure, the underlying mass matrices should have a trivial flavor structure, *i.e.* they should be proportional to the identity matrix.. This can be the case if there is an $SO(3)$ family symmetry under which the $\mathbf{10}_i^0$ and $\overline{\mathbf{5}}_i^0$ transform as triplets. The underlying Yukawa terms would then have the form

$$Y_u(\mathbf{10}_i^0 \mathbf{10}_i^0) \mathbf{5}_H + Y_d(\mathbf{10}_i^0 \overline{\mathbf{5}}_i^0) \overline{\mathbf{5}}_H + Y_\nu(\overline{\mathbf{5}}_i^0 \overline{\mathbf{5}}_i^0) \mathbf{5}_H \mathbf{5}_H / M_R. \quad (19)$$

Note that unlike Eqs. (1) and (9) there is no term here with the $\overline{45}_H$ of Higgs fields. Since all the underlying Yukawa terms must be flavor-independent, due to the $SO(3)$ symmetry, adding a term with the $\overline{45}_H$ in Eq. (19) would still leave the down quark and charged lepton mass matrices proportional to each other at the GUT scale. Therefore, the group-theoretical factors needed to avoid the “bad” minimal $SU(5)$ relation $M_d = M_\ell^T$ must appear in either the H or A matrices. In the model we are describing, they will appear in the H matrix, as will be seen.

The matrix A arises, in exactly the manner explained earlier, from the mixing of the $\overline{5}_i^0$ with the “extra” $\overline{5}'_m$, which are assumed not to transform under any flavor symmetry. Let there be at least two Standard-Models-singlet Higgs fields that are triplets under $SO(3)$, denoted by $\mathbf{1}_H^{ni}$, where n labels the Higgs triplet and i is the $SO(3)$ index. Then one can write the following mass and Yukawa terms for the fermion 5-plets:

$$\begin{aligned} & M_{mn}(\mathbf{5}'_m \overline{\mathbf{5}}'_n) + y_{mn}(\mathbf{5}'_m \overline{\mathbf{5}}_i^0) \langle \mathbf{1}_H^{ni} \rangle \\ &= \mathbf{5}'_m (M_{mn} \overline{\mathbf{5}}'_n + \Delta_{mi} \overline{\mathbf{5}}_i^0), \end{aligned} \quad (20)$$

where $\Delta_{mi} = \sum_n y_{mn} \langle \mathbf{1}_H^{ni} \rangle$. We assume that the matrices M and Δ are superheavy and of the same order. (For example, they may both be of order the GUT scale.) These terms will make N linear combinations of the $\overline{5}$ fields superheavy and leave three linear combinations light. These light linear combinations, which contain Standard Model quarks and leptons, will be denoted $\overline{5}_i$. The superheavy combinations will be denoted by $\overline{5}_m^h$.

It is easily seen that if $A \equiv [I + T^\dagger T]^{-1/2}$ and $B \equiv [I + T^\dagger T]^{-1/2} T^\dagger$, where $T = M^{-1} \Delta$, then $\overline{5}^0 = A \overline{5} + B \overline{5}^h$. Exactly as in the BC model, when substituted into Eq. (19), this leads to factors of A in the effective mass matrices of the Standard Model quarks and leptons.

The factors of H in those matrices arise, as in the BB scheme, from the mixing of $\mathbf{10}_i^0$ with the $\mathbf{10}'_A$. In order for H to come out diagonal, the $\mathbf{10}'_A + \overline{\mathbf{10}}'_A$ must transform under a flavor symmetry. A simple possibility is an abelian symmetry $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$, such that $\mathbf{10}'_A$ and $\overline{\mathbf{10}}'_A$ are odd under $Z_2^{(B)}$ if $A = B$ and even otherwise. Let there be three Standard-Model-singlet Higgs fields $\mathbf{1}_H^{Ai}$, which are triplets under $SO(3)$ and transform under $Z_2^{(1)} \times Z_2^{(2)} \times Z_2^{(3)}$ in the obvious way. Then the following mass and Yukawa terms of the fermion 10-plets are allowed

$$\overline{\mathbf{10}}'_A (Y_A \mathbf{1}_H + y_A \mathbf{24}_H) \mathbf{10}'_A + \overline{\mathbf{10}}'_A (Y'_A \mathbf{1}_H^{Ai} + y'_A \mathbf{24}_H^{Ai}) \mathbf{10}_i^0. \quad (21)$$

The role of the adjoint Higgs fields $\mathbf{24}_H$ and $\mathbf{24}_H^{Ai}$ is to introduce $SU(5)$ breaking into the quark and lepton mass matrices, through H , and thus avoid the “bad” minimal $SU(5)$ prediction that the down quark masses equal the charged lepton masses at the GUT scale. It is notationally simpler, however, to explain the mixing of the 10-plets without considering the effects of the adjoint fields in Eq. (21), so we will first discuss the unrealistic case where their VEVs are set to zero (which we will call the “minimal model”) and then later discuss the realistic case where their VEVs are non-zero.

If certain coefficients in the Higgs potential are positive then the VEVs of $\mathbf{1}_H^{Ai}$ will be orthogonal to each other in $SO(3)$ space: $\sum_i \langle \mathbf{1}_H^{Ai} \rangle \langle \mathbf{1}_H^{Bi} \rangle = c_A \delta_{AB}$. (In particular, if the coefficients of the terms $(\sum_{i=1}^3 \mathbf{1}_H^{Ai} \mathbf{1}_H^{Bi})^2$ are positive it will ensure this orthogonality.) Without loss of generality, one can then choose a basis in $SO(3)$ space such that the axes are aligned with the VEVs of the three singlet VEVs.

That is, so that $\langle \mathbf{1}_H^{Ai} \rangle = s_A \delta^{Ai}$. Defining, $Y_A \langle \mathbf{1}_H \rangle \equiv M_A$ and $Y'_A \langle \mathbf{1}_H^{Ai} \rangle \equiv \Delta_A \delta^{Ai}$, Eq. (21) with adjoint VEVs set to zero gives

$$\overline{\mathbf{10}}'_A \left(M_A \mathbf{10}'_A + \Delta_A \delta^{Ai} \mathbf{10}_i^0 \right). \quad (22)$$

The three linear combinations of 10-plets appearing with the parentheses in Eq. (22) are superheavy and will be denoted $\mathbf{10}_A^h$, whereas the three linear combinations $(-\Delta_A \mathbf{10}'_A + M_A \delta^{Ai} \mathbf{10}_i^0)$ that are orthogonal to them contain Standard Model fermions and will be denoted $\mathbf{10}_i$. This gives

$$\mathbf{10}_i^0 = \cos \theta_i \mathbf{10}_i + \sin \theta_i \delta^{Ai} \mathbf{10}_A^h, \quad (23)$$

where $\cos \theta_i \equiv \delta^{Ai} M_A / \sqrt{|M_A|^2 + |\Delta_A|^2}$ and $\sin \theta_i \equiv \Delta_A / \sqrt{|M_A|^2 + |\Delta_A|^2}$. Substituting this into Eq. (19), one finds that every factor of $\mathbf{10}_i$ in the effective Yukawa couplings of the Standard Model fermions is accompanied by a factor of $\cos \theta_i$, as in Eq. (3). We will assume a hierarchical pattern $|\Delta_1/M_1| \gg |\Delta_2/M_2| \gg 1 \gg |\Delta_3/M_3|$. Then we can define a matrix H by

$$H \equiv \begin{pmatrix} \cos \theta_1 & 0 & 0 \\ 0 & \cos \theta_2 & 0 \\ 0 & 0 & \cos \theta_3 \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \text{where } \alpha \ll \beta \ll \gamma \cong 1. \quad (24)$$

Substituting $\overline{\mathbf{5}}^0 = A \overline{\mathbf{5}} + B \overline{\mathbf{5}}^h$ and Eq. (23) into Eq. (19) and using Eq. (24), the effective mass matrices of the Standard Model quarks and leptons can then be written

$$\begin{aligned} M_u &= (H^2) \mu_u \\ M_d &= (H A) \mu_d \longrightarrow M_d = (H \mathcal{D}) A_\Delta \mu_d, \\ M_\ell &= (A^T H) \mu_d \longrightarrow M_\ell = A_\Delta^T (\mathcal{D} H) \mu_d, \\ M_\nu &= (A^T A) \mu_\nu \longrightarrow M_\nu = A_\Delta^T (\mathcal{D}^2) A_\Delta \mu_\nu, \end{aligned} \quad (25)$$

This is the basic result of the model. Other than certain overall mass scales (μ_u , μ_d , and μ_ν) all the flavor structure of the quarks and leptons is controlled by two matrices: a mixing matrix A and a hierarchy matrix H . In going to the last expressions in each line of Eq. (25), we have used $A = \mathcal{D} A_\Delta \mathcal{U}$ and absorbed \mathcal{U} by field redefinitions (as explained previously). We write the matrix \mathcal{D} as $\mathcal{D} = \text{diag}(\delta, \epsilon, 1)d$ and absorb the factors of d into redefined mass scales μ'_d and μ'_ν . One therefore ends up with the following result (for the “minimal” version of the model):

$$\begin{aligned} M_u &= \begin{pmatrix} |\alpha|^2 & 0 & 0 \\ 0 & |\beta|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mu_u, \\ M_d &= M_\ell^T = \begin{pmatrix} |\alpha\delta| & 0 & 0 \\ 0 & |\beta\epsilon| & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mu'_d, \\ M_\nu &= \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ ce^{i\theta} & a & 1 \end{pmatrix} \begin{pmatrix} \delta^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mu'_\nu. \end{aligned} \quad (26)$$

Of course, the form obtained for M_ν is the same as shown in Eq. (17). The parameters called $pe^{i\theta_p}$ and $qe^{i\theta_q}$ in Eq. (17) are here called ϵ^2 and δ^2 . It should be noted that in Eq. (26), the phases of δ , ϵ , α , and β do not matter for the matrices M_u , M_d , and M_ℓ , as they can be absorbed into the fermion fields. But for the neutrino mass matrix M_ν the phases of δ and ϵ do make a difference, and have to take definite values to fit the neutrino masses and maxing angles.

One easily sees from Eq. (26) that in this “minimal model” one has, to very good approximation, the following “postdictions”:

$$\begin{aligned} m_u : m_c : m_t &= |\alpha|^2 : |\beta|^2 : 1, \\ m_d : m_s : m_b &= m_e : m_\mu : m_\tau = |\alpha\delta| : |\beta\epsilon| : 1, \\ q^2 : p^2 : 1 &= |\delta|^2 : |\epsilon|^2 : 1. \end{aligned} \tag{27}$$

From fitting the neutrino masses and mixing angles [2], one can determine $|\epsilon| = \sqrt{p} \cong \sqrt{0.1525} = \frac{1}{2.56}$ and $|\delta| = \sqrt{q} \cong \sqrt{0.0141} = \frac{1}{8.44}$. (See the discussion after Eq. (17).) And one can obtain the values of $|\alpha|$ and $|\beta|$ directly from the up quark mass ratios: $|\beta| = \sqrt{m_c/m_t} = \frac{1}{17.8}$ and $|\alpha| = \sqrt{m_u/m_t} = \frac{1}{393}$. (We take the fermion masses here and in the following equation to be the masses at 2×10^{16} as run up to that scale using the Standard Model renormalization group equations [5].) From these values one has the following result:

$$\begin{aligned} \text{minimal model hierarchy} \quad |\alpha\delta| : |\beta\epsilon| : 1 &= \frac{1}{3,317} : \frac{1}{45.6} : 1 \\ \text{actual lepton ratios} \quad m_e : m_\mu : m_\tau &= \frac{1}{3,650} : \frac{1}{17.3} : 1 \\ \text{actual quark ratios} \quad m_d : m_s : m_b &= \frac{1}{865} : \frac{1}{45.6} : 1. \end{aligned} \tag{28}$$

One sees that the minimal model works surprisingly well, in fact better than in the BB idea taken by itself, where the inter-family mass ratios of the charged leptons and of the down quarks are $\alpha : \beta : 1$, as shown in Eq. (7). (That would give $m_e/m_\tau \sim m_d/m_b \sim \alpha \sim \frac{1}{393}$, which is off by an order of magnitude for the electron.) Thus the factors of δ and ϵ , which come from combining the BB and BC ideas, give more realistic down quark and charged lepton mass hierarchies.

The combined model we are describing (so far in a minimal form) is more explanatory than the BC model. In the BC model the inter-family mass hierarchies of the up quarks, down quarks, charged leptons, and neutrinos are completely unrelated, being determined by four diagonal matrices whose elements are free parameters. In the combined model, these hierarchies are all related, and related in a way that we have just seen is qualitatively correct. The 12 parameters in the four hierarchical diagonal matrices of the BC model are replaced by just 7 parameters in the minimal model: $|\alpha|$, $|\beta|$, $|\delta|$, $|\epsilon|$, μ_u , μ'_d , and μ'_ν . This would be a huge increase in predictivity, but of course it is *too* predictive, since the minimal model gives the “bad” minimal $SU(5)$ prediction that the charged lepton masses are equal to the down quark masses at the GUT scale. To cure this problem requires that group-theoretical factors reflecting the breaking of $SU(5)$ appear in the fermion mass matrices. The simplest way for this to happen is through the matrix H as a result of the adjoint Higgs fields in Eq. (21) getting non-zero VEVs. We shall now look at this in detail.

3 The group-theoretical factors that distinguish M_d from M_ℓ

As can be seen from Eq. (28), the group-theoretical factors must enhance the muon mass and the d quark mass by about a factor of about 3 while having little effect on the other quark and lepton masses. One obtains a similar conclusion if one runs the fermion masses assuming low-energy supersymmetry. Using the results of [5], where the masses are run up to 2×10^{16} GeV, in the MSSM with $\tan \beta = 10$, one finds

$$\begin{aligned}
(m_u, m_c, m_t) / m_t &= \left(\frac{1}{179,500}, \frac{1}{368}, 1 \right), \\
\Rightarrow (\alpha, \beta, 1) \equiv \left(\sqrt{\frac{m_u}{m_t}}, \sqrt{\frac{m_c}{m_t}}, 1 \right) &= \left(\frac{1}{424}, \frac{1}{19.2}, 1 \right), \\
\Rightarrow (|\alpha\delta|, |\beta\epsilon|, 1) &= \left(\frac{1}{3,575} : \frac{1}{49.1} : 1 \right), \\
(m_d, m_s, m_b) / m_b &= \left(\frac{1}{1142}, \frac{1}{60.14}, 1 \right), \\
(m_e, m_\mu, m_\tau) / m_b &= \left(\frac{1}{2,967}, \frac{1}{14.1}, 1.24 \right).
\end{aligned} \tag{29}$$

These imply that

$$\begin{aligned}
\left(\frac{m_d}{|\alpha\delta|}, \frac{m_s}{|\beta\epsilon|}, m_b \right) / m_b &= (3.13, 0.817, 1), \\
\left(\frac{m_e}{|\alpha\delta|}, \frac{m_\mu}{|\beta\epsilon|}, m_\tau \right) / m_b &= (1.21, 3.49, 1.24).
\end{aligned} \tag{30}$$

The ratios given in Eq. (30), which are all predicted to be equal to 1 in the minimal model, must be accounted for by the group-theoretical factors.

Seemingly, the simplest way to do this is through the coupling of adjoint Higgs fields to the 10-plets of fermions, as shown in Eq. (21). Let us first just consider the effect of the VEV of the $\mathbf{24}_H$, which couples as $\overline{\mathbf{10}}'_A(y_A \mathbf{24}_H) \mathbf{10}'_A$. If we define κ_A by $\frac{y_A \langle \mathbf{24}_H \rangle}{Y_A \langle \mathbf{1}_H \rangle} = \kappa_A Y_f / 2$, where f stands for the fermion type u, u^c, d , or ℓ^c , then the effect is that in Eq. (22), M_A gets replaced by $M_A(1 + \kappa_A Y_f / 2)$. Suppose that we assume that $|\Delta_1 / M_1| \gg |\Delta_2 / M_2| \gg |\Delta_3 / M_3| \sim 1$, then the angles defined after Eq. (23) are different for different fermion types and given approximately by

$$\begin{aligned}
\cos \theta_1^f &\cong \left| \frac{M_1}{\Delta_1} (1 + \kappa_1 Y_f / 2) \right| \\
\ll \cos \theta_2^f &\cong \left| \frac{M_2}{\Delta_2} (1 + \kappa_2 Y_f / 2) \right| \\
\ll \cos \theta_3^f &\cong \left[1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 + \kappa_3 Y_f / 2)^{-2} \right]^{-1/2} \cong 1,
\end{aligned} \tag{31}$$

where $Y_f / 2$ is the weak hypercharge of the fermion of type f . Then the matrix H defined in Eq. (24) is replaced by matrices H_f , which are different for different types of fermion in the 10-plets:

$$H_f \equiv \begin{pmatrix} \cos \theta_1^f & 0 & 0 \\ 0 & \cos \theta_2^f & 0 \\ 0 & 0 & \cos \theta_3^f \end{pmatrix}, \quad (32)$$

and the fermion mass matrices have the forms

$$\begin{aligned} M_u &= (H_u H_{u^c}) \mu_u \\ M_d &= (H_d A) \mu_d \longrightarrow M_d = (H_d \mathcal{D}) A_\Delta \mu_d, \\ M_\ell &= (A^T H_{\ell^c}) \mu_d \longrightarrow M_\ell = A_\Delta^T (\mathcal{D} H_{\ell^c}) \mu_d, \\ M_\nu &= (A^T A) \mu_\nu \longrightarrow M_\nu = A_\Delta^T (\mathcal{D}^2) A_\Delta \mu_\nu, \end{aligned} \quad (33)$$

If we consider the masses of the charged fermions of the second and third families, there are four mass ratios ($\frac{m_c}{m_t}$, $\frac{m_s}{m_b}$, $\frac{m_\mu}{m_\tau}$, and $\frac{m_\tau}{m_b}$) that must be fit using the parameters in Eq.(31), and there are four such parameters, namely $|\Delta_3/M_3|$, κ_3 , $|\Delta_2/M_2|$, and κ_2 .

Consider first the ratio m_τ/m_b . As is well-known this is predicted in minimal $SU(5)$ to be 1 at the GUT scale, as is also the case in the minimal version of the present model. In reality, however, this ratio is not exactly 1, though it is near to 1 (especially in the MSSM). In fact, for $\tan \beta = 10$ it is 1.24 at the GUT scale as shown in Eq. (30). With the group-theoretic factors of Eq. (31) one sees that it is given by

$$1.24 = \left(\frac{m_\tau}{m_b} \right)_{M_{GUT}} = \frac{\cos \theta_3^{\ell^c}}{\cos \theta_3^d} = \sqrt{\frac{1 + \left| \frac{\Delta_3}{M_3} \right|^2 \left(1 + \frac{1}{6} \kappa_3 \right)^{-2}}{1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 + \kappa_3)^{-2}}}, \quad (34)$$

which is indeed close to but not exactly 1, for $\Delta_3/M_3 < 1$. We can also write (putting in the values given in Eq. (30)):

$$0.817 = \frac{m_s/m_b}{\epsilon \sqrt{m_c/m_t}} = \sqrt{\frac{1 + \frac{1}{6} \kappa_2}{1 - \frac{2}{3} \kappa_2}} \left(\frac{1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 + \frac{1}{6} \kappa_3)^{-2}}{1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 - \frac{2}{3} \kappa_3)^{-2}} \right)^{1/4}, \quad (35)$$

and

$$3.49 = \frac{m_\mu/m_b}{\epsilon \sqrt{m_c/m_t}} = \frac{1 + \kappa_2}{\sqrt{(1 + \frac{1}{6} \kappa_2)(1 - \frac{2}{3} \kappa_2)}} \left(\frac{1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 + \frac{1}{6} \kappa_3)^{-2}}{1 + \left| \frac{\Delta_3}{M_3} \right|^2 (1 - \frac{2}{3} \kappa_3)^{-2}} \right)^{1/4}. \quad (36)$$

Eqs. (34) to (36) contain three equations with three unknowns κ_2 , κ_3 , and $|\Delta_3/M_3|$. They are solved by the values $\kappa_2 = 11.2$, $\kappa_3 = -2$, and $|\Delta_3/M_3| = 0.86$. The remaining ratio m_c/m_t can then be fit by the choice $|\Delta_2/M_2| = 110$.

Fitting the first family masses is more involved. There are three additional masses to be fit (m_e , m_d , and m_u), but the expressions in Eqs. (31) have only two additional parameters (κ_1 and $|\Delta_1/M_1|$). Indeed, it turns out that there is no fit. It is for this reason that one must include the effect of the term containing $\mathbf{24}_H^{Ai}$ in Eq. (21). Actually, only one such adjoint Higgs field is required to obtain

a good fit, namely 24_H^{1i} . However, as the expressions are somewhat complicated looking, we do not show them.

One sees, then, that introducing the group-theoretic factors required to break the well-known minimal $SU(5)$ mass degeneracies means that the model ends up with as many free parameters as there are in the BC model of [2]. Thus combining that model with the BB idea leads to no increase in the number of precise quantitative predictions. However, there is a gain in explanatory power, in that the inter-family mass hierarchies of the different types of fermions are related to each other in a way that is qualitatively correct.

4 The typical values of δ and ϵ

We now turn to a discussion of the values of δ and ϵ , the elements of the diagonal matrix \mathcal{D} . It is a non-trivial condition for the viability of the model that the same values of $|\delta|$ and $|\epsilon|$ give realistic results both for the neutrino properties and for the mass hierarchies of the down quarks and charged leptons. As we have seen, the model clears this hurdle. The fit to the neutrino properties obtained in [2] gives $|\delta| \cong \frac{1}{8.44}$ and $|\epsilon| \cong \frac{1}{2.56}$, and these values also give realistic mass hierarchies, as shown in Eq. (27).

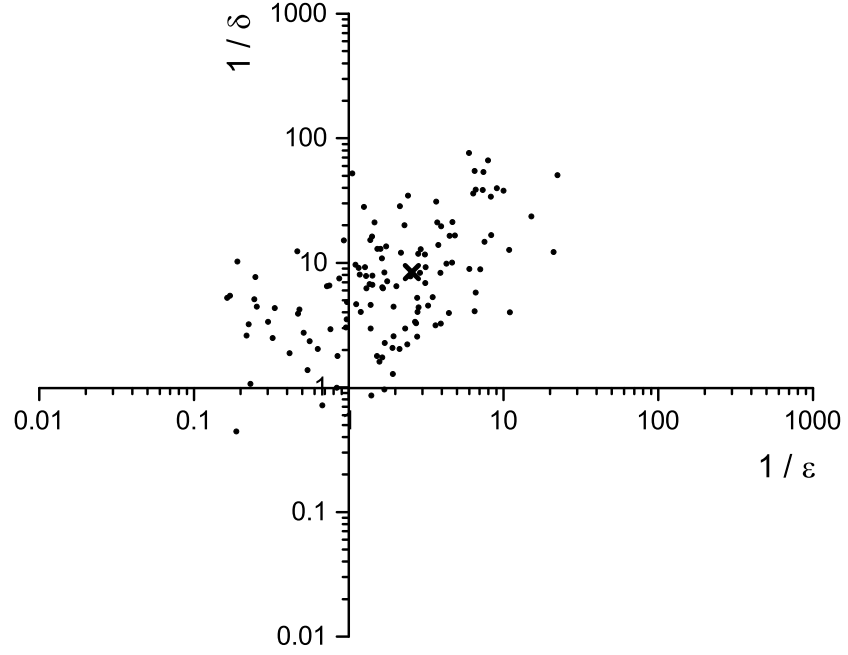
The question arises whether these are natural values for $|\delta|$ and $|\epsilon|$ to have. Why should there be any hierarchy in the elements of \mathcal{D} ? And why should that hierarchy be parallel to the hierarchy in H , with the diagonal elements increasing from the first to the third family? And why should they have these particular values? It turns out that the values of $|\delta|$ and $|\epsilon|$ needed for good fits are indeed natural, in the sense that they lie in the middle of the range of values that are most “likely” given the values of the elements of the triangular matrix A_Δ , as we will now show.

The matrix $\mathcal{D} = \text{diag}(\delta, \epsilon, 1)$ arises from bringing the matrix A to the form $A = \mathcal{D}A_\Delta\mathcal{U}$, as previously explained. The matrix A , in turn, is defined by $A \equiv (I + T^\dagger T)^{-1/2}$, where $T = M^{-1}\Delta$, and M and Δ are the matrices appearing in Eq. (20). It is natural to assume that the matrices M and Δ are both roughly of order the grand unification scale, but there is no symmetry reason why M and Δ should have any special form. Consequently, the matrix T has no reason to have any special form either.

Suppose that the elements of T are treated as random complex variables all of which have the same probability distribution. For each choice of T , one can compute the matrix A , and from that determine the matrices \mathcal{D} and A_Δ . Not surprisingly, one finds that the elements of \mathcal{D} are correlated with those of A_Δ . In fact, simple arguments show that if the elements of A_Δ that we have called a and b are large, then typically $|\delta| \sim 1/ab$ and $|\epsilon| \sim 1/a$. Since fitting the CKM angles gives $a \sim 2$ and $b \sim 4$, as can be seen from Eq. (16), the most likely values are $|\delta| \sim 1/8$ and $|\epsilon| \sim 1/2$.

This is confirmed by a numerical search treating the elements of T as random variables. We have randomly generated one million matrices T whose elements are given by $T_{ij} = 10^{r_{ij}}e^{i\theta_{ij}}$, with $-1 < r_{ij} < +1$ and $0 < \theta_{ij} < 2\pi$ with uniform probability distribution. We compute the matrices A_Δ and \mathcal{D} for each randomly generated T , and require that the parameters in A_Δ (*i.e.* a , b , c , and θ) agree with the values in Eq. (16) within experimental limits. For those that meet this requirement, we plot the values of $|\epsilon|^{-1}$ and $|\delta|^{-1}$ in Fig. 1. One sees that there indeed tends to be a mild hierarchy $|\delta| < |\epsilon| < 1$. The dark cross in Fig. 1 represents the values that give the best fit to the neutrino properties according to [2]: $(|\epsilon|^{-1}, |\delta|^{-1}) = (2.56, 8.44)$. It is apparent from Fig. 1 that these lie in the most probable range.

Fig. 1 The values of $(|\epsilon|^{-1}, |\delta|^{-1})$ that come from randomly generated matrices T that give realistic A_Δ . The dark cross represents the values that give the best fit to neutrino properties: $(|\epsilon|^{-1}, |\delta|^{-1}) = (2.56, 8.44)$.



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